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BANACH SPACES WITHOUT LOCAL UNCONDITIONAL STRUCTURE

BY

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ABSTRACT

For a large class of Banach spaces, a general construction of subspaces without local unconditional structure is presented. As an application it is shown that every Banach space of finite cotype contains either l_2 or a subspace without unconditional basis, which admits a Schauder basis. Some other interesting applications and corollaries follow.

Introduction

In this paper we present, for a large class of Banach spaces, a general construction of subspaces with a basis which have no local unconditional structure. The method works for a direct sum of several Banach spaces with bases which have certain unconditional properties. It is then applied to Banach spaces with unconditional basis, to show that if such a space X is of finite cotype and it does not contain an isomorphic copy of l_2, then X contains a subspace with a basis and without local unconditional structure. As an immediate consequence we get that if all subspaces of a Banach space X have unconditional basis then X is l_2

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saturated (i.e., every infinite-dimensional subspace of X contains a copy of l_2). In particular, if X is a homogeneous Banach space non-isomorphic to a Hilbert space (i.e., X is isomorphic to its every infinite-dimensional subspace) then X must not have an unconditional basic sequence.

We also discuss several other situations. Let us only mention here that our method provides a uniform construction of subspaces without local unconditional structure which still have Gordon–Lewis property in all L_p spaces for $1 \leq p < \infty$, $p \neq 2$, and in all p-convexified Tsirelson spaces and their duals $1 \leq p < \infty$.

The technique developed here is based on the approach first introduced by W. B. Johnson, J. Lindenstrauss and G. Schechtman in [J-L-S] for investigating the Kalton–Peck space, which was the first example of a Banach space which admits 2-dimensional unconditional decomposition but has no unconditional basis. This approach was refined by T. Ketonen in [Ke] and subsequently generalized by A. Borzyszkowski in [B], for subspaces of L_p, with $1 \leq p < 2$.

The essential idea of the approach from [J-L-S], [Ke] and [B] is summarized (and slightly generalized for our purpose) in Section 1. In the same section we also introduce all definitions and notations. Our general construction is presented in Section 2. The additional ingredient which appears here consists of an ordered sequence of partitions of natural numbers, which allows to replace some "global" arguments used before by "local" analogues. In Section 3 we prove the main application on subspaces of spaces with an unconditional basis. Other applications and corollaries are discussed in Section 4.

After this paper was sent for publication we learnt about a spectacular structural theorem just proved by W. T. Gowers. This theorem combined with our Theorem 4.2 and a result from [G-M] shows that a homogeneous Banach space is isomorphic to a Hilbert space, thus solving in the positive the so-called homogeneous space problem. More details can be found in the paper by Gowers [G].

1. Notation and preliminaries

We use the standard notation from the Banach space theory, which can be found e.g., in [L-T.1], [L-T.2] and [T], together with all terminology not explained here. In particular, the fundamental concepts of a basis and a Schauder decomposition can be found in [L-T.1], 1.a.1 and 1.g.1, respectively.

Let us only recall fundamental notions related to unconditionality.

A basis $\{e_j\}_j$ in a Banach space X is called unconditional, if there is a constant C such that for every $x = \sum_j t_j e_j \in X$ one has $\|\sum_j \varepsilon_j t_j e_j\| \le C\|x\|$, for all $\varepsilon_j = \pm 1$ for $j = 1, 2, \ldots$. The infimum of constants C is denoted by $\mathrm{unc}(\{e_j\})$. The basis is called 1-unconditional, if $\mathrm{unc}(\{e_j\}) = 1$.

A Schauder decomposition $\{Z_k\}_k$ of a Banach space X is called C-unconditional, for some constant C, if for all finite sequences $\{z_k\}$ with $z_k \in Z_k$ for all k, one has $\|\sum_k \varepsilon_k z_k\| \le C\|\sum_k z_k\|$. For a subset $K \subset \mathbb{N}$, by Y_K denote $\overline{\mathrm{span}}\,[Z_k]_{k\in K}$.

A Banach space X has local unconditional structure if there is $C \ge 1$ such for every finite-dimensional subspace $X_0 \subset X$ there exist a Banach space F with a 1-unconditional basis and operators $u_0\colon X_0 \to F$ and $w_0\colon F \to X$ such that the natural embedding $j\colon X_0 \to X$ admits a factorization $j = w_0 u_0$ and $\|u_0\|\,\|w_0\| \le C$. The infimum of constants C is denoted by $\mathrm{l.u.st}(X)$.

We will also use several more specific notation. Let F be a Banach space with a basis $\{f_l\}_l$. For a subset $A \subset \mathbb{N}$, by $F\,|_A$ we denote $\overline{\mathrm{span}}[f_l]_{l\in A}$. If F' is another space with a basis $\{f'_l\}_l$, by $I\colon F \to F'$ we denote the formal identity operator, i.e., $I(x) = \sum_l t_l f'_l$, for $x = \sum_l t_l f_l \in F$. With some abuse of notation, we will occasionally write $\|I\colon F \to F'\| = \infty$ when this operator is not bounded.

We say that a basis $\{f_l\}_l$ dominates (resp. is dominated by) $\{f'_l\}_l$, if the operator $I\colon F \to F'$ (resp. $I\colon F' \to F$) is bounded. If the bases in F and F' are fixed and they are equivalent, by $\mathrm{equiv}(F, F')$ we denote the equivalence constant

$$\mathrm{equiv}(F, F') = \|I\colon F \to F'\|\,\|I\colon F' \to F\|; \tag{1}$$

and we set $\mathrm{equiv}(F, F') = \infty$ if the bases are not equivalent.

By $D(F \oplus F')$ we denote the diagonal subspace of $F \oplus F'$, i.e., the subspace with the basis $\{(f_j + f'_j)/\|f_j + f'_j\|\}_j$; an analogous notation will be also used for a larger (but finite) number of terms.

The following proposition is a version of a fundamental criterium due to Ketonen [Ke] and Borzyszkowski [B]. Since a modification of original arguments would be rather messy, we provide a shorter direct proof.

PROPOSITION 1.1: *Let Y be a Banach space of cotype r, for some $r < \infty$, which has a Schauder decomposition $\{Z_k\}_k$, with $\dim Z_k = 2$, for $k = 1, 2, \ldots$. If Y has local unconditional structure then there exists a linear, not neccessarily bounded, operator $T\colon \mathrm{span}\,[Z_k]_k \to \mathrm{span}\,[Z_k]_k$ such that*

(i) *$T(Z_k) \subset Z_k$ for $k = 1, 2, \ldots$;*

(ii) *Whenever for some $K \subset \mathbb{N}$ and for some $C \geq 1$, the decomposition $\{Z_k\}_{k \in K}$ of Y_K is C-unconditional, then*

$$\|T|_{Y_K} : Y_K \to Y_K\| \leq C^2 \psi \text{ l.u.st}(Y), \tag{2}$$

where $\psi = \psi(r, C_r(Y))$ depends on r and the cotype r constant $C_r(Y)$ of Y only;

(iii) $\inf_\lambda \|T|_{Z_k} - \lambda I_{Z_k}\| \geq 1/8$, *for $k = 1, 2, \ldots$.*

The proof requires a fact already used in a more general form in [B]. For sake of completeness and clarity of the exposition, we sketch the proof here.

LEMMA 1.2: *Let Y be a Banach space of cotype r which has local unconditional structure, and let $q > r$. For every $\varepsilon > 0$ and every finite-dimensional subspace $Y_0 \subset Y$ there exist a Banach space E with a 1-unconditional basis which is q-concave, and operators u: $Y_0 \to E$ and w: $E \to Y$ such that the natural embedding j: $Y_0 \to Y$ admits a factorization $j = wu$ and $\|u\|\|w\| \leq (1+\varepsilon)$ l.u.st(Y). Moreover, the q-concavity constant of E satisfies $M_{(q)}(E) \leq \phi$ where $\phi = \phi(r, q, C_r(Y))$ depends on r, q and the cotype r constant of Y only.*

Proof: Given $\varepsilon > 0$ and Y_0, let F be a space with a 1-unconditional basis $\{f_i\}_i$ and let u_0: $Y_0 \to F$ and w_0: $F \to Y$ be such that $j = w_0 u_0$ and $\|w_0\|\,\|u_0\| \leq (1+\varepsilon)$ l.u.st(Y). It can be clearly assumed that F is finite-dimensional, say $\dim F = N$. Let $\{f_i^*\}_i$ be the biorthogonal functionals.

We let E to be $\mathbb{R}^N$ with the norm $\|\cdot\|_E$ defined by

$$\|(t_i)_i\|_E = \sup_{\varepsilon_i = \pm 1} \|w_0\Big(\sum_i \varepsilon_i t_i f_i\Big)\| \qquad \text{for } (t_i) \in \mathbb{R}^N.$$

We also set, $u(x) = \Big(f_i^*(u_0 x)\Big)_i$, for $x \in Y_0$ and $w\Big((t_i)_i\Big) = \sum_i t_i w_0 f_i$, for $(t_i) \in E$.

It is easy to check that $wu(x) = x$, for $x \in Y_0$ and that $\|u\| \leq \|w_0\|\,\|u_0\|$ and $\|w\| = 1$. Clearly, the standard unit vector basis is 1-unconditional in E. Using the cotype r of Y, it can be checked that E satisfies a lower r estimate with the constant $C_r(Y)$. Thus E is q-concave for every $q > r$ with the q-concavity constant $M_{(q)}(E)$ depending on q, r and $C_r(Y)$ (cf. [L-T.2] 1.f.7). ∎

Proof of Proposition 1.1: Assume that Y has the local unconditional structure. It is enough to construct a sequence of operators T_n: span$[Z_k]_k \to$ span$[Z_k]_k$, such that for every n, the operator T_n satisfies (i) and (ii') if $K_n \subset \{1,\ldots,n\}$ and the decomposition $\{Z_k\}_{k\in K_n}$ of Y_{K_n} is C-unconditional, then $\|T\,|_{Y_{K_n}}: Y_{K_n} \to Y_{K_n}\| \le C^2\psi$ l.u.st (Y);

(iii') $\inf_\lambda \|T_n\,|_{Z_k} - \lambda I_{Z_k}\| \ge 1/8$, for $k = 1,2,\ldots,n$.

Then the existence of the operator T will follow by a standard diagonal construction.

Fix n and $\varepsilon > 0$, set $q = 2r$. Let E with a 1-unconditional basis $\{e_j\}_j$ and operators $u: Y_{\{1,\ldots,n\}} \to E$ and $w: E \to Y$ be given by Lemma 1.2, such that $j = w\,u$ and $\|u\|\,\|w\| \le (1+\varepsilon)$ l.u.st (Y); moreover, E is $2r$-concave.

Let $P_k: Y \to Z_k$ be the natural projection onto Z_k, for $k = 1,2,\ldots$. For a sequence of signs $\Theta = \{\theta_j\}$, with $\theta_j = \pm 1$ for $j = 1,2,\ldots$, define $\Lambda_\Theta: E \to E$ by $\Lambda_\Theta(y) = \sum_j \theta_j t_j e_j$, for $y = \sum_j t_j e_j \in E$. Then $\|\Lambda_\Theta\| = 1$.

For every $k = 1,2,\ldots$ pick a sequence of signs Θ_k such that

$$\sup_\Theta \inf_\lambda \|P_k w \Lambda_\Theta u P_k - \lambda I_{Z_k}\| \le (4/3) \inf_\lambda \|P_k w \Lambda_{\Theta_k} u P_k - \lambda I_{Z_k}\|.$$

Define T_n: span $[Z_k]_k \to$ span $[Z_k]_k$ by

$$T_n(y) = \sum_{k=1}^n P_k w \Lambda_{\Theta_k} u P_k(y) \qquad \text{for } y = \sum_k z_k \in \text{span}\,[Z_k]_k.$$

Clearly (i) follows just from the definition of T_n. To prove (ii'), fix $K_n \subset \{1,\ldots,n\}$. Let r_k denote the Rademacher functions on $[0,1]$. Since $(E, \|\cdot\|)$ is a $2r$-concave Banach lattice with the $2r$-concavity constant depending on r and $C_r(Y)$, and also the decomposition $\{Z_k\}_{k\in K_n}$ of Y_{K_n} is C-unconditional, by Khintchine–Maurey's inequality (cf. e.g., [L-T.2], 1.d.6) we have, for $y \in Y_{K_n}$,

$$\begin{aligned}
\|T_n\,|_{Y_{K_n}}(y)\| &= \Big\|\sum_{k=1}^n P_k w \Lambda_{\Theta_k} u P_k(y)\Big\| = \Big\|\sum_{k\in K_n} P_k w \Lambda_{\Theta_k} u P_k(y)\Big\| \\
&= \Big\|\int_0^1 \Big(\sum_{k\in K_n} r_k(t) P_k\Big)\Big(\sum_{k\in K_n} r_k(t) w \Lambda_{\Theta_k} u P_k(y)\Big) dt\Big\| \\
&\le \sup_{0\le t\le 1} \Big\|\sum_{k\in K_n} r_k(t) P_k\Big\| \times \|w\| \int_0^1 \Big\|\sum_{k\in K_n} r_k(t) \Lambda_{\Theta_k} u P_k(y)\Big\| dt \\
&\le C\,M \|w\| \Big\|\Big(\sum_{k\in K_n} |\Lambda_{\Theta_k} u P_k(y)|^2\Big)^{1/2}\Big\| \\
&= C\,M \|w\| \Big\|\Big(\sum_{k\in K_n} |u P_k(y)|^2\Big)^{1/2}\Big\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \; C M^2 \|w\| \int_0^1 \Big\| \sum_{k \in K_n} r_k(t) u P_k(y) \Big\| dt \\
&\leq \; C M^2 \|w\| \, \|u\| \int_0^1 \Big\| \sum_{k \in K_n} r_k(t) P_k(y) \Big\| dt \\
&\leq \; C^2 M^2 (1+\varepsilon) \, \mathrm{l.u.st}\,(Y) \, \|y\|.
\end{aligned}
$$

The constant M depends on r and $M_{(2r)}(E)$, hence, implicitely, on r and $C_r(Y)$; so the function ψ so obtained satisfies the requirements of (ii).

To prove (iii'), fix an arbitrary $k = 1, 2, \ldots, n$. Consider the 4-dimensional space H of all linear operators on Z_k and the subspace $H_0 = \mathrm{span}\,[I_{Z_k}]$ spanned by the identity operator on Z_k. Consider the quotient space H/H_0 and for $R \in H$, let $\widetilde{R}$ be the canonical image of R in H/H_0.

Denote the biorthogonal functionals to the basis $\{e_j\}_j$ in E by $\{e_j^*\}_j$ and consider operators $R_j = P_k w(e_j^* \otimes e_j) u P_k$ on Z_k. Since $\dim R_j(Z_k) = 1 < 2$, it is easy to see that for every $j = 1, 2, \ldots$, one has

$$
\|\widetilde{R}_j\| = \inf_{\lambda} \|R_j - \lambda I_{Z_k}\| \geq (1/2)\|R_j\|.
$$

Also recall that if F is an m-dimensional space then for any vectors $\{x_j\}_j$ in F one has

$$
\sup_{\theta_j = \pm 1} \Big\| \sum_j \theta_j x_j \Big\| \geq (1/m) \sum_j \|x_j\|.
$$

This is a restatement of the estimate for the 1-summing norm of the identity on F, $\pi_1(I_F) \leq m$, and it is a simple consequence of the Auerbach lemma (cf., e.g., [T]).

So by the definition of T_n and by the choice of Θ_k and the above estimates we get

$$
\begin{aligned}
\inf_{\lambda} \|T_n|_{Z_k} - \lambda I_{Z_k}\| &\geq (3/4) \sup_{\Theta} \inf_{\lambda} \|P_k w \Lambda_\Theta u P_k - \lambda I_{Z_k}\| \\
&= (3/4) \sup_{\theta_j = \pm 1} \Big\| \sum_j \theta_j \widetilde{R}_j \Big\| \geq (1/4) \sum_j \|\widetilde{R}_j\| \\
&\geq (1/8) \sum_j \|R_j\| \geq (1/8) \| \sum_j R_j \| = (1/8)\|I_{Z_k}\| = 1/8,
\end{aligned}
$$

completing the proof. ∎

Finally let us introduce notations connected with partitions of the set of natural numbers $\mathbb{N}$, which are essential in the sequel. A subset $A \subset \mathbb{N}$ is called an interval if it is of the form $A = \{i \mid k \leq i \leq n\}$. Sets A_1 and A_2 are called consecutive if $\max A_i < \min A_j$, for $i, j = 1, 2$ and $i \neq j$. A family of mutually disjoint subsets $\Delta = \{A_m\}_m$ is a partition of $\mathbb{N}$, if $\bigcup_m A_m = \mathbb{N}$.

For a partition $\Delta = \{A_m\}_m$ of $\mathbb{N}$, by $\mathcal{L}(\Delta)$ we denote the family

$$\mathcal{L}(\Delta) = \{L \subset \mathbb{N} \mid |L \cap A_m| = 1 \text{ for } m = 1, 2, \ldots\}. \tag{3}$$

If $\Delta' = \{A'_m\}_m$ is another partition of $\mathbb{N}$, we say that $\Delta \succ \Delta'$, if there exists a partition $\mathcal{J}(\Delta', \Delta) = \{J_m\}_m$ of $\mathbb{N}$ such that

$$\min J_m < \min J_{m+1} \qquad \text{and} \qquad A'_m = \bigcup_{j \in J_m} A_j \qquad \text{for } m = 1, 2, \ldots. \tag{4}$$

In such a situation, for $m = 1, 2, \ldots$, $\mathcal{K}(A'_m, \Delta)$ denotes the family

$$\mathcal{K}(A'_m, \Delta) = \{K \subset A'_m \mid |K \cap A_j| = 1 \text{ for } j \in J_m\}. \tag{5}$$

Finally, if $\Delta_i = \{A_{i,m}\}_m$, for $i = 1, 2, \ldots$, is a sequence of partitions of $\mathbb{N}$, with $\Delta_1 \succ \cdots \succ \Delta_i \succ \cdots$, we set, for $m = 1, 2, \ldots$ and $i = 2, 3, \ldots$

$$\mathcal{K}_{i,m} = \mathcal{K}(A_{i,m}, \Delta_{i-1}). \tag{6}$$

2. General construction of subspaces without local unconditional structure

We will now present an abstract setting in which it is possible to construct spaces without local unconditional structure, but which still admit a Schauder basis. As it is quite natural, we work inside a direct sum of several Banach spaces with bases, with each basis having certain unconditional property related to some partitions of $\mathbb{N}$. The construction of a required subspace relies on an interplay between a "good" behaviour of a basis on members of the corresponding partition and a "bad" behaviour on sets which select one point from each member of the partition. (Recall that the notation $\mathcal{K}_{i,m}$ used below was introduced in (6).)

THEOREM 2.1: *Let $X = F_1 \oplus \cdots \oplus F_4$ be a direct sum of Banach spaces of cotype r, for some $r < \infty$, and let $\{f_{i,l}\}_l$ be a normalized monotone Schauder basis in F_i, for $i = 1, \ldots, 4$. Let $\Delta_1 \succ \cdots \succ \Delta_4$ be partitions of $\mathbb{N}$, $\Delta_i = \{A_{i,m}\}_m$ for $i = 1, \ldots, 4$. Assume that there is $C \geq 1$ such that for every $K \in \mathcal{K}_{i,m}$ with $i = 2, 3, 4$ and $m = 1, 2, \ldots$, the basis $\{f_{s,l}\}_{l \in K}$ in $F_s|_K$ is C-unconditional, for*

$s = 1, \ldots, 4$; moreover, there is $\widetilde{C} \geq 1$ such that for $i = 1, 2, 3$ and $m = 1, 2, \ldots$ we have

$$\|I\colon F_i\,|_{A_{i,m}} \to F_{i+1}\,|_{A_{i,m}}\| \leq \widetilde{C}. \tag{7}$$

Assume finally that one of the following conditions is satisfied:

(i) *there is a sequence $0 < \delta_m < 1$ with $\delta_m \downarrow 0$ such that for every $i = 1, 2, 3$ and $m = 1, 2, \ldots$ and every $K \in \mathcal{K}_{i+1,m}$ we have*

$$\|I\colon D(F_1 \oplus \ldots \oplus F_i)\,|_K \to F_{i+1}\,|_K\| \geq \delta_m^{-1}; \tag{8}$$

(ii) *there is a sequence $0 < \delta_m < 1$ with $\delta_m \downarrow 0$ and $\sum_m \delta_m^{1/2} = \gamma < \infty$ such that for every $i = 1, 2, 3$ and $m = 1, 2, \ldots$ and every $K \in \mathcal{K}_{i+1,m}$ we have*

$$\|I\colon F_{i+1}\,|_K \to F_i\,|_K\| \geq \delta_m^{-1}. \tag{9}$$

Then there exists a subspace Y of X without local unconditional structure, but which still admits a Schauder basis.

Remarks: 1. The space Y will be constructed to have a 2-dimensional Schauder decomposition. If the bases $\{f_{i,l}\}_l$ are unconditional, for $i = 1, \ldots, 4$, this decomposition will be unconditional.

2. Recall that a space which admits a k-dimensional unconditional decomposition has the GL-property (cf. [J-L-S]) (with the GL-constant depending on k). Therefore the subspace Y discussed in Remark 1 above has the GL-property but fails having the local unconditional structure.

Proof: We will define 2-dimensional subspaces Z_k of X which will form a Schauder decomposition of $Y = \overline{\operatorname{span}}\,[Z_k]_k$. This decomposition will be C'-unconditional on subsets associated with the partitions $\Delta_1, \ldots, \Delta_4$, for some C' depending on C. We shall use Proposition 1.1 to show that if Y had the local unconditional structure then, letting $\psi = \psi(r, C_r(X))$ to be the function defined in this proposition, we would have

$$\text{l.u.st}\,(Y) \geq \kappa\delta_t^{-\alpha} \tag{10}$$

for an arbitrary $t = 1, 2, \ldots$; in case (i) we have $\kappa > (2^{14}3^3C^4\widetilde{C}^2\psi)^{-1}$ and $\alpha = 1/3$; in case (ii) we have $\kappa > (2^{13}(1 + 4\gamma)C^3\widetilde{C}^2\psi)^{-1}$ and $\alpha = 1/2$. This is impossible, which will conclude the proof.

For $k = 1, 2, \ldots$, vectors x_k and y_k spanning Z_k will be of the form

$$\begin{aligned} x_k &= \alpha_{1,k} f_{1,k} + \cdots + \alpha_{4,k} f_{4,k}, \\ y_k &= \alpha'_{1,k} f_{1,k} + \cdots + \alpha'_{4,k} f_{4,k}, \end{aligned}$$

such that for $k = 1, 2, \ldots$ and any scalars s and t, we will have

$$(1/2)\max(|s|, |t|) \leq \|s x_k + t y_k\| \leq 4(|s| + |t|). \tag{11}$$

Set $Z_k = [x_k, y_k]$, for $k = 1, 2, \ldots$. Clearly, $\{Z_k\}_k$ is a 2-dimensional Schauder decomposition for Y, in particular Y has a basis. Moreover, for every $i = 2, 3, 4$ and $m = 1, 2, \ldots$ and every $K \in \mathcal{K}_{i,m}$, the decomposition $\{Z_k\}_{k \in K}$ is $4C$-unconditional.

Assume that Y has the local unconditional structure. Let T be an operator obtained in Proposition 1.1. In particular, T satisfies (2) for every $K \in \mathcal{K}_{i,m}$, and every $i = 2, 3, 4$ and $m = 1, 2, \ldots$.. Let

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

denote the matrix of $T|_{Z_k}$ in the basis $\{x_k, y_k\}$, for $k = 1, 2, \ldots$, i.e., we have $T(s x_k + t y_k) = (s a_k + t b_k) x_k + (s c_k + t d_k) y_k$. Comparing the operator norm of a 2×2 matrix with the l^4_∞-norm of the sequence of entries, and using (11), we get that condition (iii) of Proposition 1.1 implies that, for all $k = 1, 2, \ldots$,

$$\inf_\lambda \max(|a_k - \lambda|, |d_k - \lambda|, |b_k|, |c_k|) \geq 2^{-5} \inf_\lambda \|T|_{Z_k} - \lambda I_{Z_k}\| \geq 2^{-8}. \tag{12}$$

For the rest of the argument we consider cases (i) and (ii) separately. We start with (i). Let $\gamma_m = \delta_m^{1/3}$, for $m = 1, 2, \ldots$. For $k \in A_{4,t}$, with $t = 1, 2, \ldots$, put

$$\begin{aligned} x_k &= f_{1,k} \qquad\qquad + \gamma_t f_{3,k} + \gamma_t^2 f_{4,k} \\ y_k &= \qquad\quad f_{2,k} \qquad\qquad\quad + \gamma_t^2 f_{4,k}. \end{aligned} \tag{13}$$

Obviously, (11) is satisfied. Fix an arbitrary $t = 1, 2, \ldots$. For $i = 1, 2, 3$, let $\mathcal{M}_i = \{m \mid |A_{i,m} \subset A_{4,t}\}$. Note that (4) yields that $\min \mathcal{M}_i \geq t$ for $i = 1, 2, 3$.

For every $m \in \mathcal{M}_2$ pick $B \in \mathcal{K}_{2,m}$. By (8) we have

$$\|I\colon F_1|_B \to F_2|_B\| \geq \gamma_m^{-3};$$

on the other hand, $\|f_{1,l}\| = \|f_{2,l}\| = \|I(f_{1,l})\|$. By continuity, there exists a sequence of scalars $\{\beta_k\}_{k\in B}$ such that $\|\sum_{k\in B}\beta_k f_{1,k}\| = 1$ and $\|\sum_{k\in B}\beta_k f_{2,k}\| = \gamma_t^{-1}$. Then, by (7) and (13) we have

$$\begin{aligned}\Big\|\sum_{k\in B}\beta_k x_k\Big\| &\le \Big\|\sum_{k\in B}\beta_k f_{1,k}\Big\| + \gamma_t\Big\|\sum_{k\in B}\beta_k f_{3,k}\Big\| + \gamma_t^2\Big\|\sum_{k\in B}\beta_k f_{4,k}\Big\|\\ &\le 1 + (\gamma_t\widetilde{C} + \gamma_t^2\widetilde{C}^2)\Big\|\sum_{k\in B}\beta_k f_{2,k}\Big\| \le 3\widetilde{C}^2,\end{aligned}$$

while

$$\begin{aligned}&\Big\|T\big(\sum_{k\in B}\beta_k x_k\big)\Big\| = \Big\|\sum_{k\in B}\beta_k(a_k x_k + c_k y_k)\Big\|\\ &\ge \Big\|\sum_{k\in B}\beta_k c_k f_{2,k}\Big\| \ge C^{-1}\inf_{k\in B}|c_k|\,\Big\|\sum_{k\in B}\beta_k f_{2,k}\Big\| \ge C^{-1}\gamma_t^{-1}\inf_{k\in A_{2,m}}|c_k|.\end{aligned}$$

This implies, by (2), that for every $m\in\mathcal{M}_2$ there exists $l\in A_{2,m}$ such that $|c_l| \le 3\,4^2C^3\widetilde{C}^2\psi\,\gamma_t\,\mathrm{l.u.st}\,(Y)$. Denote the set of these l's by L_2 and observe that $L_2\in\mathcal{L}(\Delta_2)\,|_{\mathcal{M}_2}$. If we had $|c_l| > 2^{-10}$ for some $l\in L_2$, then (10) would follow. Therefore assume that $|c_l|\le 2^{-10}$ for all $l\in L_2$.

For every $m\in\mathcal{M}_3$, set $B = L_2\cap A_{3,m}$. Then $B\in\mathcal{K}_{3,m}$ and by (8) there exists a sequence $\{\beta_k\}_{k\in B}$ such that

$$\Big\|\sum_{k\in B}\beta_k\Big(\frac{f_{1,k}+f_{2,k}}{\|f_{1,k}+f_{2,k}\|}\Big)\Big\| = 1 \qquad\text{and}\qquad \Big\|\sum_{k\in B}\beta_k f_{3,k}\Big\| = \gamma_t^{-2}.$$

Observe that the basis $\{(f_{1,k}+f_{2,k})/\|f_{1,k}+f_{2,k}\|\}_{k\in B}$ is $2C$-unconditional for every $B\in\mathcal{K}_{3,m}$. Thus,

$$\Big\|\sum_{k\in B}\beta_k f_{2,k}\Big\| \le \Big\|\sum_{k\in B}\beta_k(f_{1,k}+f_{2,k})\Big\| \le 4C.$$

Hence,

$$\Big\|\sum_{k\in B}\beta_k y_k\Big\| \le \Big\|\sum_{k\in B}\beta_k f_{2,k}\Big\| + \gamma_t^2\Big\|\sum_{k\in B}\beta_k f_{4,k}\Big\| \le 4\,C + \widetilde{C},$$

and

$$\begin{aligned}\Big\|T\Big(\sum_{k\in B}\beta_k y_k\Big)\Big\| &= \Big\|\sum_{k\in B}\beta_k(b_k x_k + d_k y_k)\Big\|\\ &\ge \gamma_t\Big\|\sum_{k\in B}\beta_k b_k f_{3,k}\Big\| \ge C^{-1}\gamma_t^{-1}\inf_{k\in L_2\cap A_{3,m}}|b_k|.\end{aligned}$$

Therefore, using (2) again, for every $m \in \mathcal{M}_3$ pick $l \in L_2 \cap A_{3,m}$ such that $|b_l| \le 4^2(4C + \widetilde{C})C^3\psi\gamma_t$ l.u.st (Y). Denote the set of these l's by L_3 and assume as before that $|b_l| \le 2^{-10}$ for all $l \in L_3$. Moreover, $L_3 \subset L_2$ and $L_3 \in \mathcal{L}(\Delta_3)\,|_{\mathcal{M}_3}$.

Finally, consider $K = L_3 \cap A_{4,t} \in \mathcal{K}_{4,t}$ and pick a sequence $\{\beta_k\}_{k\in K}$ such that

$$\Big\|\sum_{k\in K}\beta_k\Big(\frac{f_{1,k}+f_{2,k}+f_{3,k}}{\|f_{1,k}+f_{2,k}+f_{3,k}\|}\Big)\Big\| = 1 \qquad \text{and} \qquad \Big\|\sum_{k\in K}\beta_k f_{4,k}\Big\| = \gamma_t^{-3}.$$

Since $\{(f_{1,k}+f_{2,k}+f_{3,k})/\|f_{1,k}+f_{2,k}+f_{3,k}\|\}_{k\in K}$ is $3C$-unconditional, for every $K \in \mathcal{K}_{4,t}$, we have, for $i = 1,2,3$,

$$\Big\|\sum_{k\in K}\beta_k f_{i,k}\Big\| \le \Big\|\sum_{k\in K}\beta_k(f_{1,k}+f_{2,k}+f_{3,k})\Big\| \le 3^2C.$$

Thus,

$$\Big\|\sum_{k\in K}\beta_k(x_k-y_k)\Big\| \le \Big\|\sum_{k\in K}\beta_k f_{1,k}\Big\| + \Big\|\sum_{k\in K}\beta_k f_{2,k}\Big\| + \gamma_t\Big\|\sum_{k\in K}\beta_k f_{3,k}\Big\| \le 3^3\,C.$$

Moreover, since $|c_k| \le 2^{-10}$ and $|b_k| \le 2^{-10}$ for $k \in L_3$, by (12) we have

$$|a_k - b_k + c_k - d_k| \ge 2^{-9} \qquad \text{for} \quad k \in L_3. \tag{14}$$

Therefore

$$\begin{aligned}\Big\|T\Big(\sum_{k\in K}\beta_k(x_k-y_k)\Big)\Big\| &= \Big\|\sum_{k\in K}\beta_k\Big((a_k-b_k)x_k + (c_k-d_k)y_k\Big)\Big\| \\ &\ge \gamma_t^2\Big\|\sum_{k\in K}\beta_k((a_k-b_k)+(c_k-d_k))f_{4,k}\Big\| \\ &\ge C^{-1}2^{-9}\gamma_t^{-1}.\end{aligned}$$

Using (2) once more we get $3^3 4^2 C^3\psi$ l.u.st $(Y) \ge C^{-1}2^{-9}\gamma_t^{-1}$, which implies (10). This completes the proof of case (i).

In case (ii) the proof is very similar and let us describe necessary modifications. Set $\gamma_m = \delta_m^{1/2}$ for $m = 1,2,\ldots$. For $k = 1,2,\ldots$ and $k \in A_{2,m} \cap A_{3,s}$, for some $m = 1,2,\ldots$ and $s = 1,2,\ldots$, set

$$\begin{aligned} x_k &= \qquad\quad \gamma_s f_{2,k} + f_{3,k} + f_{4,k} \\ y_k &= \gamma_m f_{1,k} \qquad\qquad + f_{3,k}. \end{aligned} \tag{15}$$

Again, (11) is satisfied. Fix an arbitrary $t = 1,2,\ldots$, and define $\mathcal{M}_i$, for $i = 1,2,3$ as before. Using the fact that $\|I\colon F_2\,|_K \to F_1\,|_K\| \ge \gamma_m^{-2}$, for every

$K \in \mathcal{K}_{2,m}$ and every $m \in \mathcal{M}_2$, one can show, using (7) and (2) in a similar way as before, that there is a set $L_2 = \{l_m\}_{m \in \mathcal{M}_2} \in \mathcal{L}(\Delta_2)\,|_{\mathcal{M}_2}$ such that

$$
|c_{l_m}| \leq 3\,4^2 C^3 \widetilde{C}^2 \psi \gamma_m \text{ l.u.st}\,(Y) \qquad \text{for } m \in \mathcal{M}_2. \tag{16}
$$

One can additionally assume that $|c_{l_m}| \leq 2^{-10}$ for all $m \in \mathcal{M}_2$, otherwise, since $\min \mathcal{M}_2 \geq t$ implies $\gamma_m \leq \gamma_t$, we would immediately get (10) with $\alpha = 1/2$.

Now for every $s \in \mathcal{M}_3$ consider the set $B = L_2 \cap A_{3,s} \in \mathcal{K}_{3,s}$, and pick a sequence $\{\beta_k\}_{k \in B}$ such that

$$
\Big\|\sum_{k \in B} \beta_k f_{3,k}\Big\| = 1 \qquad \text{and} \qquad \Big\|\sum_{k \in B} \beta_k f_{2,k}\Big\| \geq \gamma_s^{-2}.
$$

If $\mathcal{M}_{2,s}$ denotes the set of indices $m \in \mathcal{M}_2$ such that $l_m \in L_2 \cap A_{3,s} = B$, then

$$
\Big\|\sum_{k \in B} \beta_k y_k\Big\| \leq \sum_{m \in \mathcal{M}_{2,s}} \gamma_m |\beta_{l_m}| + 1 \leq 2\gamma + 1, \tag{17}
$$

where the first term in the estimate is obtained by first using the triangle inequality and then using the fact that since $\{f_{3,l_m}\}_{m \in \mathcal{M}_{2,s}}$ is a monotone basic sequence, then $|\beta_{l_m}| \leq 2$ for all $l_m \in B$.

We also have

$$
\begin{aligned}
\Big\|T\Big(\sum_{k \in B} \beta_k y_k\Big)\Big\| &\geq \gamma_s \Big\|\sum_{k \in B} \beta_k b_k f_{2,k}\Big\| \\
&\geq C^{-1} \gamma_s \inf_{k \in B} |b_k| \Big\| \sum_{k \in L_2 \cap A_{3,s}} \beta_k f_{2,k}\Big\| \\
&\geq C^{-1} \gamma_s^{-1} \inf_{k \in L_2 \cap A_{3,s}} |b_k|.
\end{aligned}
$$

Thus there exists a set $L_3 \in \mathcal{L}(\Delta_3)\,|_{\mathcal{M}_3}$, $L_3 = \{l'_s\}_{s \in \mathcal{M}_3}$, such that $L_3 \subset L_2$ and

$$
|b_{l'_s}| \leq 4^2(2\gamma + 1) C^3 \psi \gamma_s \text{ l.u.st}\,(Y) \qquad \text{for } s \in \mathcal{M}_3; \tag{18}
$$

and since $\min \mathcal{M}_3 \geq t$ implies $\gamma_s \leq \gamma_t$, one can additionally assume that $|b_{l'_s}| \leq 2^{-10}$, for all $s \in \mathcal{M}_3$.

Finally set $K = L_3 \cap A_{4,t} \in \mathcal{K}_{4,t}$. Pick a sequence $\{\beta_k\}_{k \in K}$ such that $\|\sum_{k \in K} \beta_k f_{4,k}\| = 1$ and $\|\sum_{k \in K} \beta_k f_{3,k}\| \geq \gamma_t^{-2}$. Then, by the triangle inequality and by the monotonicity of the basis $\{f_{4,k}\}_k$ we get, similarly as in (17),

$$
\Big\|\sum_{k \in K} \beta_k (x_k - y_k)\Big\| \leq 2 \sum_{m \in \mathcal{M}_2} \gamma_m + 2 \sum_{s \in \mathcal{M}_3} \gamma_s + \Big\|\sum_{k \in K} \beta_k f_{4,k}\Big\| \leq 1 + 4\gamma.
$$

On the other hand, by (12), (16) and (18) we again have (14). Thus

$$\begin{aligned}\Big\|T\Big(\sum_{k\in K}\beta_k(x_k-y_k)\Big)\Big\| &\geq \Big\|\sum_{k\in K}\beta_k(a_k-b_k)+(c_k-d_k)f_{3,k}\Big\|\\ &\geq C^{-1}2^{-9}\Big\|\sum_{k\in K}\beta_k f_{3,k}\Big\| \geq C^{-1}2^{-9}\gamma_t^{-2}.\end{aligned}$$

Using (2) we get $\text{l.u.st}(Y) \geq (2^{13}(1+4\gamma)C^3\psi)^{-1}\gamma_t^{-2}$, hence (10) follows, completing the proof of case (ii). ∎

3. Subspaces of spaces with unconditional basis

Our main application of the construction of Theorem 2.1 is the following result on subspaces of spaces with unconditional basis.

THEOREM 3.1: *Let X be a Banach space with an unconditional basis and of cotype r, for some $r<\infty$. If X does not contain a subspace isomorphic to l_2 then there exists a subspace Y of X without local unconditional structure, which admits a Schauder basis.*

In particular, every Banach space of cotype r, for some $r<\infty$, contains either l_2 or a subspace without unconditional basis.

We present now the proof of the theorem, leaving corollaries and further applications to the next section.

The argument is based on a construction, for a given Banach space X, of a direct sum inside X of subspaces F_i of X, and of partitions Δ_i of $\mathbb{N}$ such that Theorem 2.1 can be applied. This construction requires several steps.

The first lemma is a simple generalization to finite-dimensional lattices of the fact that the Rademacher functions in L_p are equivalent to the standard unit vector basis in l_2.

LEMMA 3.2: *Let E be an N-dimensional Banach space with a 1-unconditional basis $\{e_j\}_j$ and for $2\leq r<\infty$ let $C_r(E)$ denote the cotype r constant of E. If $m\leq \log_2 N$ then there exist normalized vectors $f_1,\ldots,f_m$ in E, of the form*

$$f_l=\sum_j \varepsilon_j^{(l)}\alpha_j e_j \qquad \text{for } l=1,\ldots,m, \tag{19}$$

for some sequence of scalars $\{\alpha_j\}$ and $\varepsilon_j^{(l)}=\pm1$ for $l=1,\ldots,m$ and $j=1,\ldots,N$; and such that

$$\text{equiv}\left(\text{span}\,[f_l],\, l_2^m\right)\leq C, \tag{20}$$

where C depends on r and on the cotype r constant of E.

Proof: Since E is a discrete Banach lattice, the cotype r assumption implies that E is q-concave, for any $q > r$ (cf. [L-T.2]). Setting e.g., $q = 2r$, the q-concavity constant of E depends on r and $C_r(E)$. By a lattice renorming we may and will assume that this constant is equal to 1 (cf. [L-T.2] 1.d.8); the general case will follow by adjusting C.

For $1 \le p < \infty$, let $\|\cdot\|_{L_p}$ be the norm defined on $\mathbb{R}^N$ by $\|t\|_{L_p} = (N^{-1}\sum_{j=1}^N |t_j|^p)^{1/p}$, for $t = (t_j) \in \mathbb{R}^N$. It is well known consequence of Lozanovski's theorem (see [T], 39.2 and 39.3 for a related result) that there exist $\alpha_j > 0$, $j = 1,\dots,N$, such that

$$\|t\|_{L_1} \le \|\sum_{j=1}^N \alpha_j t_j e_j\| \le \|t\|_{L_q} \qquad \text{for } t = (t_j) \in \mathbb{R}^N. \tag{21}$$

Fix an integer $m \le \log_2 N$. By Khintchine's inequality there exist vectors $r_l = \{r_l(j)\}_{j=1}^N$, with $r_l(j) = \pm 1$ for $j = 1,\dots,N$, $l = 1,\dots,m$, such that for every $(b_l) \in \mathbb{R}^m$ we have

$$2^{-1/2}(\sum_{l=1}^m |b_l|^2)^{1/2} \le \|\sum_{l=1}^m b_l r_l\|_{L_1} \le \|\sum_{l=1}^m b_l r_l\|_{L_q} \le C_q(\sum_{l=1}^m |b_l|^2)^{1/2}. \tag{22}$$

Setting $f_l = \sum_{j=1}^N r_l(j)\alpha_j e_j$, for $l = 1,\dots,m$, we get, by (21),

$$\|\sum_{l=1}^m b_l r_l\|_{L_1} \le \|\sum_{l=1}^m b_l f_l\| = \|\sum_{j=1}^N \alpha_j(\sum_{l=1}^m b_l r_l(j))e_j\| \le \|\sum_{l=1}^m b_l r_l\|_{L_q},$$

for every $(b_l) \in \mathbb{R}^m$. This combined with (22) completes the required estimate. ∎

Remark: As it was pointed out to us by B. Maurey, Lemma 3.2 could be replaced by the contruction of L. Tzafriri [Tz], which implies the existence of a function $\varphi(N)$, with $\varphi(N) \to \infty$ as $N \to \infty$, such that for $m \le \varphi(N)$ every N-dimensional space E as in the lemma contains normalized vectors $f_1,\dots,f_m$ satisfying (20), which are of the form $f_l = \alpha\sum_j \pm e_j$, with an appropriate constant α.

The next proposition is the key for our argument. To simplify the statement, let us introduce one more notation. Given a partition $\Delta = \{A_m\}_m$ of $\mathbb{N}$ into consecutive intervals and a space F with a normalized Schauder basis $\{f_l\}_l$ and $C \ge 1$, we call a pair $\{\Delta, F\}$ C-regular, if the following conditions are satisfied:

(i) equiv $\left(F|_{A_m}, l_2^{|A_m|}\right) \le C$ for $m = 1, 2, \ldots$;

(ii) for every $L \in \mathcal{L}(\Delta)$, the basis $\{f_l\}_{l\in L}$ in $F|_L$ is 1-unconditional (here $\mathcal{L}(\Delta)$ is as in (3));

(iii) for arbitrary $L, L' \in \mathcal{L}(\Delta)$ one has equiv $\left(F|_L, F|_{L'}\right) = 1$.

Observe that condition (iii) means that if $L = \{l_m\}_m$, $L' = \{l'_m\}_m$, with $l_m, l'_m \in A_m$ for $m = 1, 2, \ldots$, then for every sequence of scalars (b_m) one has

$$\Big\|\sum_m b_m f_{l_m}\Big\| = \Big\|\sum_m b_m f_{l'_m}\Big\|. \tag{23}$$

PROPOSITION 3.3: *Let $E_1, E_2 \ldots$ be Banach spaces of cotype r, for some $r < \infty$. Let $\{e_{i,j}\}_j$ be a 1-unconditional basis in E_i, and assume that no sequence of disjointly supported vectors in $E_1 \oplus \cdots \oplus E_i$ is equivalent to the standard unit vector basis in l_2, for $i = 1, 2, \ldots$. Then there exists C, depending on r and the cotype r constants of E_i, such that there exist subspaces $F_i \subset E_i$ with normalized Schauder bases $\{f_{i,l}\}_l$, and partitions $\Delta_i = \{A_{i,m}\}_m$ of $\mathbb{N}$ into consecutive intervals, for $i = 1, 2, \ldots$, with $\Delta_1 \succ \Delta_2 \succ \cdots$, satisfying the following: for each $i = 1, 2, \ldots$ $\{\Delta_i, F_i\}$ is C-regular and one of the following mutually exclusive conditions is satisfied: either for every $L \in \mathcal{L}(\Delta_i)$ one has*

$$\|I\colon l_2 \to F_i|_L\| = \infty, \tag{24}$$

or for every $L \in \mathcal{L}(\Delta_i)$ one has

$$\|I\colon l_2 \to F_i|_L\| < \infty. \tag{25}$$

Furthermore, one also has

(iv) *If (24) holds for some i, then the partition $\Delta_{i+1} = \{A_{i+1,m}\}_m$ satisfies*

$$\inf_m \inf\left\{2^{-3m}\|I\colon l_2^{|K|} \to F_i|_K\| \;\Big|\; K \in \mathcal{K}_{i+1,m}\right\} \ge C. \tag{26}$$

On the other hand, let M denote the set (which may be empty) of all $s \in \mathbb{N}$ such that for every $L \in \mathcal{L}(\Delta_s)$ one has $\|I\colon l_2 \to F_s|_L\| < \infty$. If $i \in M$, put $M_i = M \cap \{1, \ldots, i\}$; then the partition $\Delta_{i+1} = \{A_{i+1,m}\}_m$ satisfies

$$\inf_m \inf\left\{2^{-3m}\|I\colon D\Big(\sum_{s\in M_i} \oplus F_s\Big)|_K \to l_2^{|K|}\| \;\Big|\; K \in \mathcal{K}_{i+1,m}\right\} \ge C. \tag{27}$$

Proof: In the first part of the proof we show that given space E of cotype r with a 1-unconditional basis $\{e_j\}_j$, and a partition $\Delta = \{A_m\}_m$ of $\mathbb{N}$ into consecutive intervals, there exists a subspace $F \subset E$ with a normalized Schauder basis $\{f_l\}_l$ such that $\{\Delta, F\}$ is C-regular, for an appropriate constant C, and that either (24) or (25) is satisfied for every $L \in \mathcal{L}(\Delta)$.

For an arbitrary $m = 1, 2, \ldots$, let $k_m = |A_m|$ and let $E^{(m)} = \operatorname{span}\{e_j \mid 2^{k_m} < j \le 2^{k_m+1}\}$. Since $\dim E^{(m)} \ge 2^{k_m}$, by Lemma 3.2 there exist vectors $f_l \in E^{(m)}$, for $l \in A_m$, such that

$$\operatorname{equiv}\left(\operatorname{span}[f_l]_{l \in A_m}, l_2^{k_m}\right) \le C; \tag{28}$$

and there is a sequence $\{\alpha_j\}$ of real numbers such that the f_l's are of the form

$$f_l = \sum_{j=2^{k_m}+1}^{2^{k_m+1}} \pm \alpha_j e_j \qquad \text{for } l \in A_m, \quad m = 1, 2, \ldots. \tag{29}$$

We let $F = \operatorname{span}[f_l]_l$. Then (i) is implied by (28). Next observe that f_l and $f_{l'}$ have consecutive supports, whenever $l \in A_m$ and $l' \in A_{m'}$ and $m \ne m'$. This and (28) easily yield that $\{f_l\}_l$ is a Schauder basis in F. Also, $\{f_l\}_{l \in L}$ is a 1-unconditional basis in $F|_L$, for every $L \in \mathcal{L}(\Delta)$, which shows (ii).

By (29) we get that if (b_m) is a scalar sequence then for every $L = \{l_m\}_m \in \mathcal{L}(\Delta)$, the vector $\sum_m b_m f_{l_m}$ is of the form

$$\sum_m b_m \sum_{j=2^{k_m}+1}^{2^{k_m+1}} \pm \alpha_j e_j;$$

a specific choice of the l_m's which constitute the set L effects only the choice of the signs in the inner summation. Since the basis $\{e_j\}$ is 1-unconditional, (23) follows, hence (iii) follows as well.

Finally observe that for a fixed $L \in \mathcal{L}(\Delta)$, exactly one of conditions (24) and (25) holds. Moreover, by (iii), the norms of the formal identity operators involved do not depend on a choice of the set $L \in \mathcal{L}(\Delta)$.

We now pass to the second part of the proof, the inductive construction of Δ_i's and F_i's, which ensures also condition (iv). Let $A_{1,m} = \{m\}$ for $m = 1, 2, \ldots$ and let $\Delta_1 = \{A_{1,m}\}_m$.

Assume that $i \ge 1$ and that partitions $\Delta_1 \succ \cdots \succ \Delta_i$ and subspaces $F_1, \ldots, F_{i-1}$ have been constructed to satisfy conditions (i)–(iv). Let $F_i \subset E_i$ be a subspace constructed in the first part of the proof for $\Delta = \Delta_i$. The construction of Δ_{i+1} depends on which of two, (24) or (25), holds for F_i.

Assume first that (24) holds and fix an arbitrary set $L \in \mathcal{L}(\Delta_i)$. Enumerate $L = \{l_j\}_j$ with $l_j \in A_{i,j}$ for $j = 1, 2, \ldots$. There exist $1 = j_0 < j_1 < \cdots < j_m < \cdots$ such that if $J_m = \{j_{m-1} \le j < j_m\}$, then

$$\|I\colon l_2^{|J_m|} \to F_i\,|_{L\,|_{J_m}}\| \ge C^2 2^{3m} \qquad \text{for } m = 1, 2, \ldots. \tag{30}$$

We then set

$$A_{i+1,m} = \bigcup_{j \in J_m} A_{i,j} \qquad \text{for } m = 1, 2, \ldots. \tag{31}$$

By (23) and (30) it is clear that (26) is satisfied in this case.

Assume now that (25) holds, so $i \in M$. There is a constant C' such that for all $s \in M_i$ the estimate $\|I\colon l_2 \to F_s\,|_L\| < C'$ holds for all $L \in \mathcal{L}(\Delta_s)$; hence also for all $L \in \mathcal{L}(\Delta_i)$, since sets from $\mathcal{L}(\Delta_i)$ are subsets of sets from $\mathcal{L}(\Delta_s)$, for every $s < i$. Fix an arbitrary $L \in \mathcal{L}(\Delta_i)$. We then have

$$\|I\colon l_2 \to D\Big(\sum_{s \in M_i} \oplus F_s\Big)\,|_L\| < |M_i|\, C'.$$

Note that if $l, l' \in L \in \mathcal{L}(\Delta_i)$ and $l \ne l'$ then $f_{s,l}$ and $f_{s,l'}$ have consecutive supports, hence $\{f_{s,l}\}_{l \in L}$ forms a block basis of $\{e_{s,j}\}_j$, for $s \in M_i$. Therefore by our assumptions, the basis $\{\sum_{s \in M_i} f_{s,l}\}_{l \in L}$ in $D(\sum_{s \in M_i} \oplus F_s)$ is not equivalent to the standard unit vector basis in l_2. Thus

$$\|I\colon D\Big(\sum_{s \in M_i} \oplus F_s\Big)\,|_L \to l_2\| = \infty. \tag{32}$$

Now the construction of a partition Δ_{i+1} satisfying (27) is done by formulas completely analogous to (30) and (31), in which the use of (24) is replaced by (32). ∎

Finally, the proof of the main result follows formally from Proposition 3.3.

Proof of Theorem 3.1: Write X as an unconditional sum $X = \sum_i \oplus E_i$, of 13 spaces E_i, each with a 1-unconditional basis $\{e_{i,j}\}_j$. Let $\Delta_1 \succ \cdots \succ \Delta_{13}$ be partitions of $\mathbb{N}$ and $F_i \subset E_i$ be subspaces with Schauder bases $\{f_{i,l}\}_l$, constructed in Proposition 3.3. Renorming the spaces F_i if necessary, we may assume that the bases $\{f_{i,l}\}_l$ are monotone.

Now the C-regularity properties imply all the preliminary assumptions of Theorem 2.1, including (7). To prove the remaining conditions (i) or (ii) observe that either there exist four consecutive spaces $\{F_{i_k}\}_k$ satisfying (25), or (24) holds for some three (not necessarily consecutive) spaces $\{F_{i_k}\}_k$.

In either case, we let $\Lambda_k = \Delta_{i_k}$ and $F'_k = F_{i_k}$, for $k = 1, \ldots, 4$ (in the latter case we set $i_4 = i_3 + 1$).

It is easy to check that in the former case, (25) yields (27), while in the latter case (24) yields (26). Thus the remaining assumptions of Theorem 2.1 are satisfied with $\delta_m = 2^{-3m}$, which concludes the proof. ∎

4. Corollaries and further applications

Recall a still open question whether a Banach space whose all subspaces have an unconditional basis is isomorphic to a Hilbert space. From results on the approximation property by Enflo, Davie, Figiel and Szankowski, combined with Maurey–Pisier–Krivine theorem, it follows that such a space X has, for every $\varepsilon > 0$, cotype $2+\varepsilon$ and type $2-\varepsilon$ (cf. e.g., [L-T.2], 1.g.6). Theorem 3.1 obviously implies that X has a much stronger property: its every infinite-dimensional subspace contains an isomorphic copy of l_2. A space X with this property is called l_2-saturated.

THEOREM 4.1: *Let X be an infinite-dimensional Banach space whose all subspaces have an unconditional basis. Then X is l_2-saturated.*

Another well known open problem, going back to Mazur and Banach, concerns so-called homogeneous spaces. An infinite-dimensional Banach space is called homogeneous if it is isomorphic to each of its infinite-dimensional subspaces. The question is whether every homogeneous Banach space is isomorphic to a Hilbert space. The same general argument as before shows that a homogeneous space X has cotype $2+\varepsilon$ and type $2-\varepsilon$, for every $\varepsilon > 0$. W. B. Johnson showed in [J] that if both X and X^* are homogeneous and X has the Gordon–Lewis property, then X is isomorphic to a Hilbert space. More information about homogeneous spaces the reader can find in [C]. The following obvious corollary removes the assumption on X^*, however it requires a stronger property of X itself.

THEOREM 4.2: *If a homogeneous Banach space X contains an infinite unconditional basic sequence then X is isomorphic to a Hilbert space.*

Let us recall here that it was believed for a long time that every Banach space might contain an infinite unconditional basic sequence. This conjecture was disproved only recently by W. T. Gowers and B. Maurey in [G-M], who actually constructed a whole class of Banach spaces failing this and related properties.

Let us now discuss some easy consequences of the main construction, which might be of independent interest.

COROLLARY 4.3: *Let $X = F_1 \oplus \cdots \oplus F_4$ be a direct sum of Banach spaces of cotype r, for some $r < \infty$, and assume that F_i has a 1-unconditional basis $\{f_{i,l}\}_l$. for $i = 1, \ldots, 4$. Assume that the basis $\{f_{i,l}\}_l$ dominates $\{f_{i+1,l}\}_l$, and that no subsequence of $\{f_{i,l}\}_l$ is equivalent to the corresponding subsequence of $\{f_{i+1,l}\}_l$, for $i = 1, 2, 3$. Then there exists a subspace Y of X without local unconditional structure, which admits an unconditional decomposition into 2-dimensional spaces.*

Proof: Let $\Delta_1 \succ \cdots \succ \Delta_4$ be arbitrary partitions of $\mathbb{N}$ into infinite subsets $\{A_{i,m}\}$. The domination assumption implies (7). On the other hand, the second assumption allows for a construction of partitions which also satisfy (9). Hence the conclusion follows from Theorem 2.1 and Remark 2 above. ∎

Remark: In fact, Corollary 4.3 can be proved directly from Proposition 1.1. To define x_k and y_k spanning Z_k, let $\Lambda_2 = \{B_m\}_m$ be any partition of $\mathbb{N}$ into infinite sets and write each B_m as a union $B_m = \bigcup_n B_{m,n}$ of an infinite number of infinite sets $B_{m,n}$. (Using the natural enumeration of $\mathbb{N} \times \mathbb{N}$, we get this way a partition $\Lambda_1 = \{B_{m,n}\}_{m,n}$ with $\Lambda_1 \succ \Lambda_2$.) Then for $k \in B_{m,n}$, with $m, n = 1, 2, \ldots$ put

$$\begin{aligned} x_k &= \phantom{2^{-m-n}e_{1,k}} \quad 2^{-m}e_{2,k} \quad +e_{3,k} \quad +e_{4,k} \\ y_k &= 2^{-m-n}e_{1,k} \qquad\qquad\qquad +e_{3,k}. \end{aligned}$$

The rest of the proof is the same as in case (ii) of Theorem 2.1.

If $\{x_i\}$ is a basic sequence in a Banach space X, and $1 \le p < \infty$, we say that l_p is crudely finitely sequence representable in $\{x_i\}$ if there is a constant $C \ge 1$ such that for every n there is a subset $B_n \subset \mathbb{N}$ such that $\{x_i\}_{i \in B_n}$ is C-equivalent to the unit vector basis in l_p^n.

COROLLARY 4.4: *Let X be a Banach space of cotype r, for some $r < \infty$, and with a 1-unconditional basis $\{e_l\}_l$; let $1 \le p < \infty$. Assume that no sequence $\{x_j\}_j$ of disjointly supported vectors of the form $x_j = \sum_{l \in L_j} e_l$, where $|L_j| \le 3$ for $j = 1, 2, \ldots$, is equivalent to the unit vector basis of l_p. Moreover assume that X has one of the following properties:*

(i) *l_p is crudely finitely sequence representable in $\{e_l\}_l$, and the basis $\{e_l\}_l$ either is dominated by or dominates the standard unit vector basis in l_p;*

(ii) *l_p is crudely finitely sequence representable in every subsequence of $\{e_l\}_l$. Then X contains a subspace Y without local unconditional structure, which admits a 2-dimensional unconditional decomposition.*

Proof: First observe a general fact concerning a basis $\{e_l\}_l$ whose no subsequence is dominated by the standard unit vector basis in l_p. An easy diagonal argument shows that if a partition $\Delta = \{A_j\}_j$ of $\mathbb{N}$ into finite sets is given then for an arbitrary M and every $j_0 \in \mathbb{N}$ there is $j_1 > j_0$ such that for any set $K \subset \mathbb{N}$ such that $|K| = j_1 - j_0$ and $|K \cap A_j| = 1$ for $j_0 < j \leq j_1$, one has $\|I\colon l_p^{|K|} \to E\,|_K\| \geq M$. In particular, given constant C, there exists a partition $\Delta' = \{A'_m\}_m$ of $\mathbb{N}$, with $\Delta \succ \Delta'$ such that for every $m = 1, 2, \ldots$ and for every $K \in \mathcal{K}(A'_m, \Delta)$ one has

$$\|I\colon l_p^{|K|} \to E\,|_K\| \geq C\, 2^{3m}. \tag{33}$$

Now, in case (i), write X as a direct sum $E_1 \oplus \cdots \oplus E_4$, such that each E_i has a 1-unconditional basis $\{e_{i,l}\}_l$. Assume that the basis $\{e_l\}_l$ dominates the basis in l_p, hence so does every basis $\{e_{i,l}\}_l$. Using the general observation above, we can define by induction partitions $\Delta_1 \succ \cdots \succ \Delta_4$ and subsequences $\{f_{i,j}\}_j$ of $\{e_{i,l}\}_l$, so that for all k and all $A = A_{k,m} \in \Delta_k$, sequences $\{f_{i_k,j}\}_{j\in A}$ are C-equivalent to the standard unit vector basis in $l_p^{|A|}$, and at the same time, the spaces $\operatorname{span}[f_{i_k,j}]_{j\in K}$, with $K \in \mathcal{K}_{k+1,m}$, satisfy the lower estimate (33). Thus (9) is satisfied (with $\delta_m = 2^{-3m}$).

If the basis $\{e_l\}_l$ is dominated by the basis in l_p, so is every basis $\{e_{i,l}\}_l$, and also all bases in $D(E_1 \oplus \cdots \oplus E_i)$, for $i = 1, 2, 3$. An analogous argument as before, which additionally requires the assumption on sequences $\{x_j\}$, leads to a construction of partitions satisfying (8). Then the existence of the subspace Y follows from Theorem 2.1 and Remark 1 after its statement.

In case (ii), write $X = E_1 \oplus \cdots \oplus E_7$. By passing to subsequences we get that for each i, each subsequence of the basis $\{e_{i,l}\}_l$, either is dominated by or dominates the standard unit vector basis in l_p, for $i = 1, \ldots, 7$. Therefore there is a set $I = \{i_1, \ldots, i_4\}$ such that for all $i \in I$, the bases $\{e_{i,l}\}_l$ have the same, either former or latter, domination property. Then the proof can be concluded the same way as in case (i). ∎

For $1 \leq q < \infty$, the space $L_q([0,1])$ contains a subspace isomorphic to $X = (\sum_n \oplus l_2^n)_q$, which, for $q \neq 2$, satisfies the assumptions of Corollary 4.4 (i) for $p = 2$. Therefore $L_q([0,1])$ contains a subspace without local unconditional structure but which admits a 2-dimensional unconditional decomposition. By Remark 2 in Section 2, this subspace has the Gordon-Lewis (GL-) property. For $1 \leq q < 2$, this gives a somewhat more elementary proof of Ketonen's result [Ke]. For $2 < q < \infty$ the construction seems to be new. Ketonen's result could be also derived from Corollary 4.3 by noticing that in this case the space $L_q([0,1])$ contains a subspace isometric to $(l_{q_1} \oplus \cdots \oplus l_{q_4})_q$, for $1 \leq q \leq q_1 < \cdots < q_4 < 2$ (cf. e.g., [L-T.2], 2.f.5).

Corollary 4.4 can also be applied to construct subspaces without local unconditional structure in p-convexified Tsirelson spaces $T_{(p)}$ and in their duals. This solves the question left open in [K]. The spaces $T_{(2)}$ and $T^*_{(2)}$ provide the most important examples of so-called weak Hilbert spaces, and they were discussed in [P]. For general p and notably for $p = 1$, these spaces were presented in detail in [C-S]. First construction of a weak Hilbert space without unconditional basis was given by R. Komorowski in [K] by a method preceeding the technique presented here.

COROLLARY 4.5: *The p-convexified Tsirelson space $T_{(p)}$, for $1 \leq p < \infty$, and the dual Tsirelson $T^*_{(p)}$, for $1 < p < \infty$, contain subspaces without local unconditional structure, but which admit 2-dimensional unconditional decomposition; in particular they have the Gordon–Lewis property.*

Proof: The spaces $T_{(p)}$ and $T^*_{(p)}$ satisfy the assumptions of Corollary 4.4, both (i) and (ii), for p and p', respectively. ∎

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